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NAVAL POSTGRADUATE SCHOOL MONTEREY CA R N FORREST
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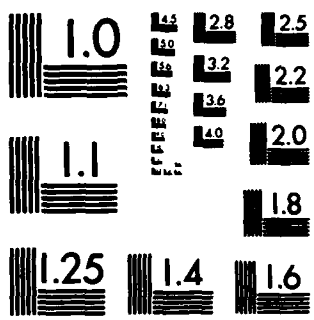
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NAVAL POSTGRADUATE SCHOOL

Monterey, California

PROGRAMS FOR A TARGET
POSITION ESTIMATION PROCEDURE

BY

R. N. FORREST

March 1983

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The report contains program listings and user instructions for an HP-41CV, a Sharp PC-1500 (or TRS-80 PC-2), a Sharp PC-1211 (or TRS-80 PC-1), a Casio FX-702P and a TI-59. The programs implement a bearings-only position estimation procedure. A development for the procedure is included in the report.

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The programs in this report are for use within the Department of the Navy, and they are presented without representation or warranty of any kind.

TABLE OF CONTENTS

I. Introduction

This report contains a target position estimation program for each of the following calculators: The Hewlett-Packard HP-41CV, the Sharp PC-1500 (or Radio Shack TRS-80 PC-2), the Sharp PC-1211 (or Radio Shack TRS-80 PC-1), the Casio FX-702P and the Texas Instruments TI-59.

The programs provide a means of implementing a position estimation procedure that is described in Appendix 1. The procedure is based on the following assumptions: Bearings taken on or from a target are available from two or more stations of known position. The positions of the stations and the target are such that they can be considered to be on the surface of a plane (a flat earth model). The error in a bearing taken on or from a station is a normal random variable. Its standard deviation (bearing error) is known and its mean (bias) is zero (if there is bias in a bearing, it is known and it is removed). The errors in bearing measurements are independent.

The position estimation procedure requires an initial estimate of the target's position. In the programs that are given here, the initial estimate is at the intersection of the bearing lines determined by the first two target bearings that are input to the program. Because of the use of this method to choose an initial estimate, the first two stations, with respect to the order of data input, should be chosen so that the intersection of their bearing lines is likely to be closer to the target's position than the intersection of the bearing lines from any other pair of stations.

The target ranges and errors of the bearings of the first two stations determine the distance between the intersection of their bearing lines and the target's position. In particular, if the angular separation between two stations as seen from the target is small relative to the bearing error of one or both of the stations, the bearing lines from the two stations may not intersect. If they do not intersect and they are not parallel, their reciprocal bearing lines will. In this case, if the observed bearings from the two stations were the first two bearing inputs to the program, the initial estimate of the target's position would be at the intersection of the two reciprocal bearing lines, and a gross error in the final position estimate could result.

II. Program User Instructions

Before using a program for the first time, refer to the notes that follow the user's instructions. Also, note the comments in Section I regarding the relationship between the accuracy of the computed estimates and the order of data entry.

For a PC-1211, use the PC-1500 User Instructions. Use the DEF mode and substitute SHFT for DEF wherever DEF appears in the instructions.

Station positions are determined with respect to a known reference point. The reference point can be a station position, in this case the station bearing and range from the reference point are both zero.

Program listings are given in Section IV and two example applications are discussed in Section III.

Position Estimation Program

HP-41CV User Instructions

Step	Instruction	Prompt	Press
1	Select USER mode and enter the program (see Note 1). Press Σ+ to run the program.		Σ+
2	Key in, in decimal degrees, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 2).	OBS BRG?	R/S
3	Key in the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	STA BRG?	R/S
4	Key in the station range (in any units) from the reference point. Use zero if the station is the reference point.	STA RNG?	R/S
5	Key in the bearing error (standard deviation) in decimal degrees.		R/S
6	Repeat Steps 2, 3, 4 & 5 for one or more additional stations.		
7	Compute bearing & range estimates.		1/x
8	A computed target bearing estimate in decimal degrees is displayed.	BRG=est.	R/S
9	A computed target range estimate in meters. For an elliptical containment region, go to Step 10 or Step 17. To enter additional data from new or old stations, go to Step 24.	RNG=est.	
10	For a containment ellipse of a given containment probability, press LN.		LN
11	Key in the desired containment probability.	PRB?	R/S
12	Computed value of the ellipse size: (See Note 3.)	SIZE=val.	R/S

Step	Instruction	Prompt	Press
13	Computed semi-major axis length:	SMJ=val.	R/S
14	Computed major axis direction:	DRC=val.	R/S
15	Computed semi-minor axis length:	SMI=val.	R/S
16	Computed containment ellipse area: (See Note 4.)	A = val.	
17	For a containment ellipse of a given size, press LOG.		LOG
18	Key in the desired containment ellipse size (see Note 3).	SIZE?	R/S
19	Computed value of the containment probability:	PRB=val.	R/S
20	Computed semi-major axis length:	SMJ=val.	R/S
21	Computed major axis direction:	DRC=val.	R/S
22	Computed semi-minor axis length:	SMI=val.	R/S
23	Computed containment ellipse area: (See Note 4.)	A = val.	
24	To enter additional data from new or old stations, press $\sqrt{x}$ and then repeat Steps 2, 3, 4 & 5		$\sqrt{x}$

Notes:

1. The program size is 35. The key assignments are: TPE → Σ+, CON → √x, EST → 1/x, SIZ → LOG and PRB → LN. If they are not present, they must be made in order to follow the user instructions. For an alternative, see Appendix 2.
2. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).
3. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis = $k\sigma_{MJ}$ and semi-minor axis = $k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.
4. The area units are the range units squared.

Position Estimation Program

PC-1500 User Instructions

Step	Instruction	Prompt	Press
1	Enter the program. To run the program, press DEF, A. For a PC-1211, see Note 5.		DEF A
2	Key in, in decimal degrees, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 1).	OBS BRG?	ENTER
3	Key in the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	STA BRG?	ENTER
4	Key in the station range (in any units) from the reference point. Use zero if the station is the reference point.	STA RNG?	ENTER
5	Key in the bearing error (standard deviation) in decimal degrees.	BRG ERR?	ENTER
6	Repeat Steps 2, 3, 4 & 5 for one or more additional stations.		
7	Compute bearing & range estimates.		DEF Z
8	A computed target bearing estimate in decimal degrees is displayed.	BRG=est.	ENTER
9	A computed target range estimate is displayed. For an elliptical containment region, go to Step 10 or Step 17. To enter additional data from new or old stations, go to Step 24.	RNG=est.	
10	For a containment ellipse of a given containment probability, press DEF, X.		DEF X
11	Key in the desired containment probability.	PRB?	ENTER
12	Computed value of the ellipse size: See Note 2.	SIZE=val.	ENTER

Step	Instruction	Prompt	Press
13	Computed semi-major axis length:	SMJ=val.	ENTER
14	Computed major axis direction:	DRC=val.	ENTER
15	Computed semi-minor axis length:	SMI=val.	ENTER
16	Computed containment ellipse area: (See Note 3.)	A = val.	
17	For a containment ellipse of a given size, press DEF, S.		DEF S
18	Key in the desired containment ellipse size (see Note 2).	SIZE?	ENTER
19	Computed value of the containment probability:	PRB=val.	ENTER
20	Computed semi-major axis length:	SMJ=val.	ENTER
21	Computed major axis direction:	DRC=val.	ENTER
22	Computed semi-minor axis length:	SMI=val.	ENTER
23	Computed containment ellipse area: (See Note 3.)	A = val.	
24	To enter additional data from new or old stations, press DEF, C and then repeat Steps 2, 3, 4 & 5.		DEF C

Notes:

1. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).
2. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis = $k\sigma_{MJ}$ and semi-minor axis = $k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.
3. The area units are the range units squared.
4. For a definition of the program initiating keys and their function, press DEF, H. The display will show: TPE = A EST = Z SIZ = S PRB = X. Next press ENTER. The display will show: CON = C. To repeat the display, press ENTER.
5. For a PC-1211, use the DEF mode and substitute SHFT for DEF wherever DEF appears in the instructions.

Position Estimation Program

FX-702P User Instructions

Step	Instruction	Prompt	Press
1	Enter the program (see Note 1). To run the program, first press F1, Ø.		F1 Ø
2	Next, key in 1 and press EXE.	OPTION?	1 EXE
3	Key in, in decimal degress, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 2).	OBS BRG?	EXE
4	Key in the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	STA BRG?	EXE
5	Key in the station range (in any units) from the reference point. Use zero if the station is the reference point.	STA RNG?	EXE
6	Key in the bearing error (standard deviation) in decimal degrees.	BRG ERR?	EXE
7	Repeat Steps 3, 4, 5 & 6 for one or more additional stations.		
8	To compute bearing & range estimates, first press F1, Ø.		F1 Ø
9	Next, key in 2 and press EXE.	OPTION?	2 EXE
10	A computed target bearing estimate in decimal degrees is displayed.	BRG=est.	CONT
11	A computed target range estimate is displayed. For an elliptical containment region, go to Step 12 or Step 20. To enter additional data from new or old stations, go to Step 28.	RNG=est.	
12	For a containment ellipse of a given containment probability, first press F1, Ø.		F1 Ø
13	Next, key in 3 and press EXE.	OPTION?	3 EXE

Step	Instruction	Prompt	Press
14	Key in the desired probability.	PRB?	EXE
15	Computed value of the ellipse size: See Note 3.	SIZE=val.	CONT
16	Computed semi-major axis length:	SMJ=val.	CONT
17	Computed major axis direction:	DRC-val.	CONT
18	Computed semi-minor axis length:	SMI=val.	CONT
19	Computed containment ellipse area: See Note 4.	A = val.	
20	For a containment ellipse of a given size, first press Fl, Ø.		Fl Ø
21	Next, key in 4 and press EXE.	OPTION?	4 EXE
22	Key in the desired containment ellipse size (see Note 3).	SIZE?	EXE
23	Computed value of the containment probability:	PRB=val.	CONT
24	Computed semi-major axis length:	SMJ=val.	CONT
25	Computed major axis direction:	DRC=val.	CONT
26	Computed semi-minor axis length:	SMI=val.	CONT
27	Computed containment ellipse area: See Note 4.	A = val.	
28	To enter additional data from new or old stations, first press Fl, Ø.		Fl Ø
29	Next, key in 5, press EXE and then repeat Steps 3, 4, 5 & 6.	OPTION?	5 EXE

Notes:

1. Enter the program in PØ for F1 Ø activation. Before running the program, first press F2 then DEFM then 1 and then EXE. This is required in order to use the array variables in the program.
2. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).
3. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis = $k\sigma_{MJ}$ and semi-minor axis = $k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.
4. The area units are the range units squared.
5. For a definition of the program options and their function, press F1, Ø. Then after OPTION? is displayed, enter Ø and press EXE. The display will show: TPE = 1 EST = 2 SIZ = 3. Next, press CONT. The display will show: PRB = 4 CON = 5. To repeat the displays, press CONT.

POSITION ESTIMATION PROGRAM

TI-59 USER INSTRUCTIONS

Step	Instruction	Enter	Press	Display
1	Enter the program (see Note 1).			
2	To run the program, press A.		A	9
3	Enter, in decimal degrees, the observed bearing of the target from a station or the reciprocal of the observed bearing of the station from the target (see Note 2).	θ	B	θ
4	Enter the station bearing in decimal degrees from the reference point. Use zero if the station is the reference point.	α	R/S	α
5	Enter the station range from the reference point. Use zero if the station is the reference point. Use any units.	ρ	R/S	ρ
6	Enter the bearing error (standard deviation) in decimal degrees. After pressing R/S, the display indicates the order number of the data entry.	e	R/S	n
7	Repeat Steps 3, 4, 5 & 6 for at least one more station.			
8	Display a bearing estimate $\hat{\phi}$ with respect to the reference station.		C	$\hat{\phi}$
9	Display a range estimate $\hat{r}$ with respect to the reference station. For an elliptical containment region, go to Step 10 or Step 15. To enter additional data from new or old stations, repeat Steps 3, 4, 5 & 6.		R/S	$\hat{r}$
10	For a containment ellipse of a given containment probability, enter the containment probability. Next, press E and display the ellipse size (see Note 3).	p	E	k
11	Display the semi-major axis length.		R/S	$k\sigma_{MJ}$
12	Display the major axis direction.		R/S	γ
13	Display the semi-minor axis length.		R/S	$k\sigma_{MI}$

Step	Instruction	Enter	Press	Display
14	Display the containment ellipse area. (See Note 4.)		R/S	area
15	For a containment ellipse of a given size, enter the containment ellipse size (See Note 3). Next, press D and display the containment probability.	k	D	p
16	Display the semi-major axis length.		R/S	$k\sigma_{MJ}$
17	Display the major axis direction.		R/S	γ
18	Display the semi-minor axis length.		R/S	$k\sigma_{MI}$
19	Display the containment ellipse area. (See Note 4.)		R/S	area
20	To enter additional data from new or old stations, repeat Steps 3, 4, 5 & 6.			

Notes:

1. The program requires the normal partition. If the calculator has been in use, this can be assured by turning the calculator off and then on before loading the program.
2. Reciprocal bearings are used when bearings are taken on known positions from the unknown position (target).
3. In the model that is the basis for the program, for a given probability of containment, the minimum area containment region is an ellipse centered on the position estimate. The semi-major axis = $k\sigma_{MJ}$ and semi-minor axis = $k\sigma_{MI}$ where k is the size of the ellipse and σ_{MJ} and σ_{MI} are the standard deviations (uncertainty measure) of the position estimate in the major axis and minor axis directions.
4. The area units are the range units squared.
5. Negative bearing estimates and negative major axis directions can result. To convert a negative bearing estimate to the value that would be output by the other programs, add 360° to the estimate. For example, -5° becomes 355°. To convert a negative direction, add 180°. For example, -5° becomes 175°.

III. Two Examples

In Scenario 1, the scenario for the first example, bearings are taken on a target from three separate stations. Figure 1 on Page 18 shows Scenario 1 and Table 1 below gives the station data. The stations are numbered according to the order of station data input to the program.

TABLE 1

	OBS BRG	STA BRG	STA RNG	BRG ERR
Station 1	038°	334°	13500	4°
Station 2	324°	050°	11350	3°
Station 3	003°	000°	0	4°

Note from Table 1 that the reference point is at Station 3.

Program outputs for Scenario 1 are given in List 1 on Page 19. List 1 and List 2 which gives program outputs for Scenario 2, are copies of printer tapes that were generated with a Casio FP-10 printer and a Casio FX-702P using the Casio FX-702P program. With allowance for differences in user instructions, display formats and round-off errors, the tapes indicate the output that should be obtained using any of the other calculator programs except for the TI-59 program where equivalent negative angles occur.

In Scenario 2, the scenario for the second example, bearings are taken from a target on three separate stations. Figure 2 on Page 20 shows Scenario 2 and Table 2 below gives the station data. As in Scenario 1, the stations are numbered according to the order of station data input to the program. In this scenario the reciprocal of the observed bearings are used in order to provide an equvalent scenario that is appropriate for the program.

TABLE 2

	OBS BRG	RCP BRG	STA BRG	STA RNG	BRG ERR
Station 1	211°	031°	000°	0	3°
Station 2	172°	352°	115°	11100	3°
Station 3	146°	326°	082°	13800	3°

Note from Table 2 that the reference point is at Station 1.

Program outputs for Scenario 2 are given in List 2 on Page 21. The two bearing estimates in the list are the reciprocal of the target's estimate of the bearing of Station 1.

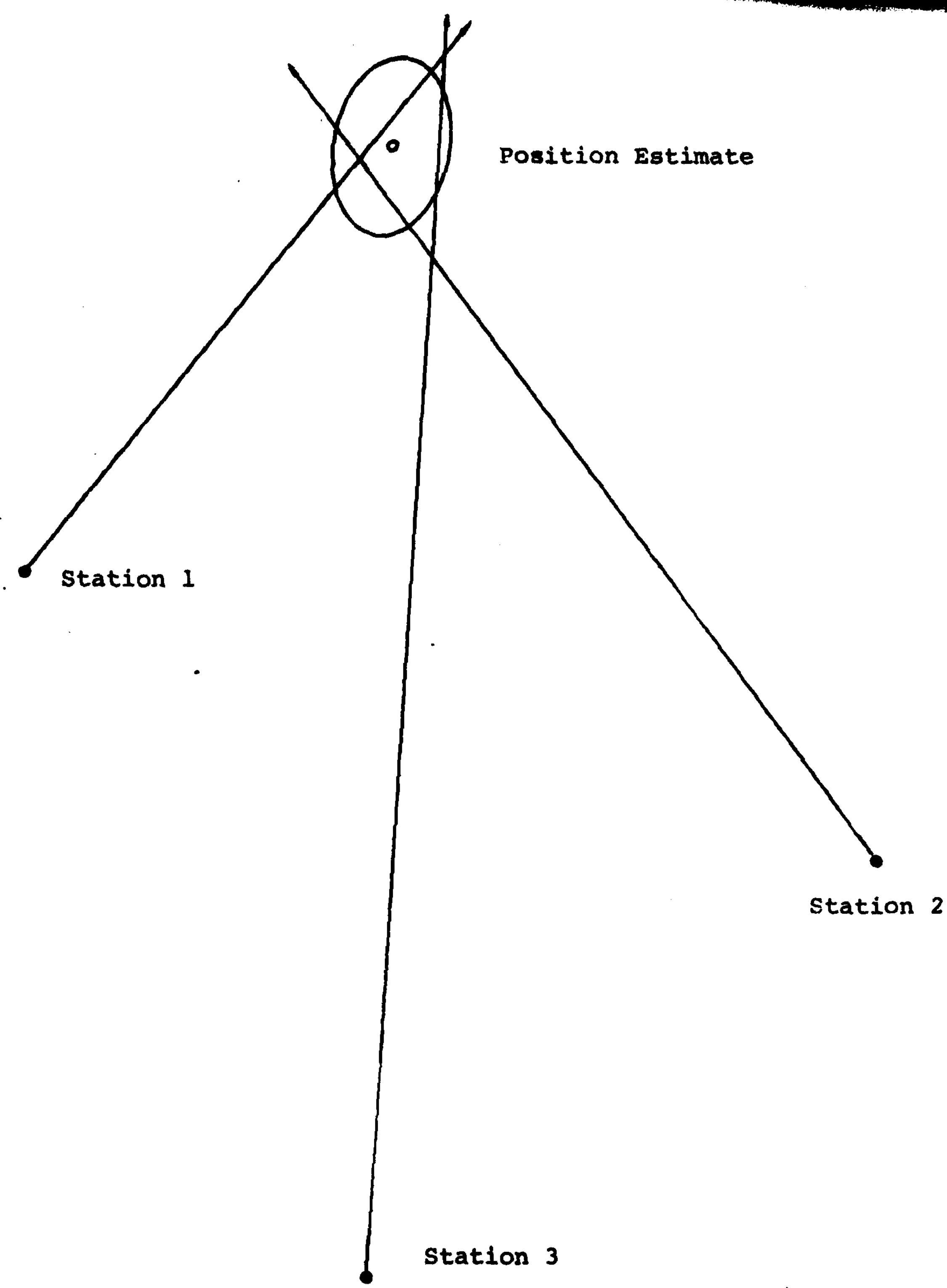

Figure 1. A position estimate and a .83 confidence region. Table 1 gives the bearings of the indicated bearing lines and List 1 gives the bearing and the range of the position estimate from Station 3.

LIST 1

```
OPTION?
0
TPE=1 EST=2 SIZ=3

PRB=4 CON=5

OPTION?
1
OBS BRG?
38
STA BRG?
334
STA RNG?
13500
BRG ERR?
4
OBS BRG?
324
STA BRG?
50
STA RNG?
11350
BRG ERR?
3
OBS BRG?

OPTION?
2
BRG= 359.51

RNG= 19494.39

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 1737.32

DIR= 17.69

SMI= 1232.96

A= 6729444.91

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 1864.12

DIR= 17.69

SMI= 1322.94

A= 7747559.77

END

OPTION?
5
OBS BRG?
3
STA BRG?
0
STA RNG?
0
BRG ERR?
4
OBS BRG?

OPTION?
2
BRG= 0.15

RNG= 19553.70

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 1712.95

DIR= 12.48

SMI= 1129.90

A= 6080418.40

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 1837.97

DIR= 12.48

SMI= 1212.36

A= 7000340.38

END
```

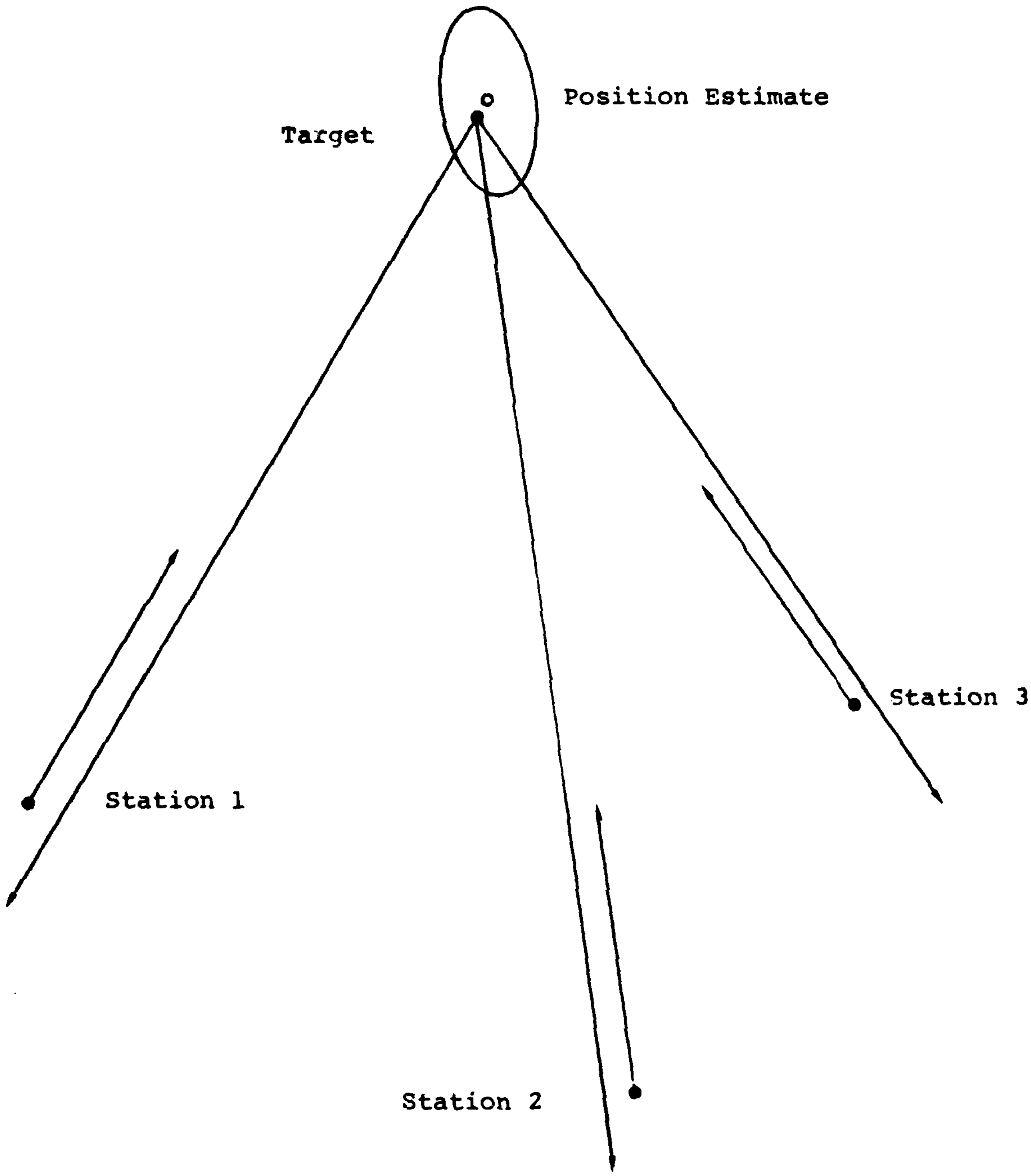

Figure 2. A position estimate and .71 confidence region. Table 2 gives the bearings of the indicated bearing lines and List 2 gives the reciprocal bearing and the range of the position estimate from Station 1.

List 2

```
OPTION?
0
TPE=1 EST=2 SIZ=3

PRB=4 CON=5

OPTION?
1
OBS BRG?
31
STA BRG?
0
STA RNG?
0
BRG ERR?
3
OBS BRG?
352
STA BRG?
115
STA RNG?
11100
BRG ERR?
3
OBS BRG?

OPTION?
2
BRG= 31.00

RNG= 14792.53

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 3606.56

DIR= 15.39

SMI= 1253.72

A= 14205070.91

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 3869.77

DIR= 15.39

SMI= 1345.22

A= 16354192.26

END

OPTION?
5
OBS BRG?
326
STA BRG?
82
STA RNG?
13800
BRG ERR?
3
OBS BRG?

OPTION?
2
BRG= 31.65

RNG= 13765.30

END

OPTION?
3
SIZE?
2
PRB= 0.86

SMJ= 2023.19

DIR= 172.06

SMI= 967.90

A= 6151983.32

END

OPTION?
4
PRB?
.9
SIZE= 2.15

SMJ= 2170.85

DIR= 172.06

SMI= 1038.54

A= 7082732.54

END
```

IV. Program Listings

HP-41CV PROGRAM

```
01♦LBL "TPE"
02 DEG
03 CLRG
04 .001
05 STO 09
06 26
07 STO 10
08♦LBL "CON"
09 FIX 0
10 "OBS BRG?"
11 PROMPT
12 STO 11
13 "STA BRG?"
14 PROMPT
15 STO 12
16 "STA RNG?"
17 PROMPT
18 STO 13
19 "BRG ERR?"
20 PROMPT
21 STO 14
22 1
23 ST+ 08
24 2
25 RCL 08
26 X>Y?
27 GTO 02
28 1
29 ST+ 10
30 RCL 11
31 STO IND 10
32 1
33 ST+ 10
34 RCL 12
35 STO IND 10
36 1
37 ST+ 10
38 RCL 13
39 STO IND 10
40 1
41 ST+ 10
42 RCL 14
43 STO IND 10
44 1
45 RCL 08
46 X=Y?
47 GTO "CON"
48 RCL 29
49 RCL 28
50 RCL 27
51 -
52 SIN
53 *
54 STO 15
55 RCL 33
56 RCL 32
57 RCL 31
58 -
59 SIN
60 *
61 STO 16
62 RCL 31
63 RCL 27
64 -
65 SIN
66 STO 17
67 X=0?
68 GTO 08
69 RCL 15
70 RCL 31
71 SIN
72 *
73 RCL 16
74 RCL 27
75 SIN
76 *
77 -
78 RCL 17
79 /
80 STO 01
81 RCL 15
82 RCL 31
83 COS
84 *
85 RCL 16
86 RCL 27
87 COS
88 *
89 -
90 RCL 17
91 /
92 STO 02
93 26
94 STO 10
95♦LBL 05
96 1
97 ST+ 10
98 RCL IND 10
99 STO 11
100 1
101 ST+ 10
102 RCL IND 10
103 STO 12
104 1
105 ST+ 10
106 RCL IND 10
107 STO 13
108 1
109 ST+ 10
110 RCL IND 10
111 STO 14
112 XEQ 07
113 ISG 09
114 GTO 05
115 GTO "CON"
116♦LBL 02
117 XEQ 07
118 GTO "CON"
119♦LBL 07
120 RCL 01
121 RCL 13
122 RCL 12
123 SIN
124 *
125 -
126 RCL 02
127 RCL 13
128 RCL 12
129 COS
130 *
131 -
132 R-P
133 STO 16
134 X<>Y
135 STO 15
136 RCL 11
137 RCL 15
138 -
139 180
140 X<>Y
141 X<=Y?
142 GTO 11
143 360
144 -
145♦LBL 11
146 RCL 14
147 /
148 STO 17
149 RCL 16
150 RCL 14
```

```
151 *
152 PI
153 *
154 180
155 /
156 STO 18
157 RCL 15
158 COS
159 RCL 18
160 /
161 STO 19
162 RCL 15
163 SIN
164 RCL 18
165 /
166 STO 20
167 RCL 19
168 X↑2
169 ST+ 03
170 RCL 19
171 RCL 20
172 *
173 ST+ 04
174 RCL 20
175 X↑2
176 ST+ 05
177 RCL 17
178 RCL 19
179 *
180 ST+ 06
181 RCL 17
182 RCL 20
183 *
184 ST+ 07
185 RTN
186♦LBL "EST"
187 FIX 2
188 RCL 04
189 X↑2
190 RCL 03
191 RCL 05
192 *
193 -
194 STO 20
195 X=0?
196 GTO 00
197 RCL 01
198 RCL 04
199 RCL 07
200 *
201 RCL 05
202 RCL 06
203 *
204 -
205 RCL 20
206 /
207 +
208 STO 26
209 RCL 02
210 RCL 03
211 RCL 07
212 *
213 RCL 04
214 RCL 06
215 *
216 -
217 RCL 20
218 /
219 +
220 RCL 26
221 X<>Y
222 R-P
223 STO 21
224 X<>Y
225 X>0?
226 GTO 06
227 360
228 +
229♦LBL 06
230 STO 22
231 RCL 04
232 RCL 03
233 RCL 05
234 X≠Y?
235 GTO 03
236 RCL 04
237 SIGN
238 45
239 *
240 GTO 04
241♦LBL 03
242 -
243 /
244 2
245 *
246 ATAN
247 2
248 /
249♦LBL 04
250 STO 23
251 COS
252 X↑2
253 RCL 05
254 *
255 RCL 23
256 1
257 P-R
258 *
259 2
260 *
261 RCL 04
262 *
263 STO 26
264 -
265 RCL 23
266 SIN
267 X↑2
268 RCL 03
269 *
270 +
271 RCL 20
272 CHS
273 /
274 STO 24
275 RCL 23
276 SIN
277 X↑2
278 RCL 05
279 *
280 RCL 26
281 +
282 RCL 23
283 COS
284 X↑2
285 RCL 03
286 *
287 +
288 RCL 20
289 CHS
290 /
291 STO 25
292 RCL 24
293 X<=Y?
294 GTO 01
295 STO 25
296 X<>Y
297 STO 24
298 90
299 ST+ 23
300♦LBL 01
```

```
301 "BRG="
302 ARCL 22
303 AVIEW
304 STOP
305 "RNG="
306 ARCL 21
307 AVIEW
308 STOP
309 "END"
310 GTO 00
311+LBL "SIZ"
312 "SIZE"
313 PROMPT
314 STO 11
315 X↑2
316 2
317 /
318 CHS
319 E↑X
320 1
321 X<>Y
322 -
323 STO 00
324 "PRB="
325 ARCL 00
326 AVIEW
327 STOP
328 GTO 09
329+LBL "PRB"
330 "PRB?"
331 PROMPT
332 CHS
333 1
334 +
335 LN
336 2
337 *
338 CHS
339 SQRT
340 STO 11
341 "SIZE="
342 ARCL 11
343 AVIEW
344 STOP
345+LBL 09
346 RCL 23
347 STO 12
348 RCL 11
349 RCL 25
350 SQRT
351 *
352 RCL 11
353 RCL 24
354 SQRT
355 *
356 X<>Y
357 "SMJ="
358 ARCL X
359 AVIEW
360 STOP
361 RCL 23
362 X>0?
363 GTO 10
364 180
365 +
366+LBL 10
367 "BRC="
368 ARCL X
369 AVIEW
370 STOP
371 "SMI="
372 RDN
373 ARCL Y
374 AVIEW
375 STOP
376 PI
377 *
378 *
379 "A="
380 ARCL X
381+LBL 00
382 AVIEW
383 STOP
384 "END"
385 GTO 00
386+LBL 08
387 AVIEW
388 STOP
389 "NO SOL"
390 GTO 08
391 END
```

```
10:"A":CLEAR :DIM
   A(7):DEGREE
15:"C":INPUT "OBS
    BRG? ";P:
   INPUT "STA BRG
   ? ";Q:INPUT "S
   TA RNG? ";R:
   INPUT "BRG ERR
   ? ";O:PAUSE "
   "
20:IF I=2GOTO 50
25:I=I+1:A(I-1)=P
   :A(I+1)=Q:A(I+
   3)=R:A(I+5)=O:
   IF I=1GOTO 15
30:X=A(4)*SIN (A(
   2)-A(0)):Y=A(5
   )*SIN (A(3)-A(
   1)):Z=SIN (A(1
   )-A(0)):IF Z=0
   GOTO 150
35:U=(X*SIN A(1)-
   Y*SIN A(0))/Z:
   V=(X*COS A(1)-
   Y*COS A(0))/Z
40:FOR M=0TO 1
45:P=A(M):Q=A(M+2
   ):R=A(M+4):O=A
   (M+6):GOSUB 17
   0:NEXT M:GOTO
   15
50:GOSUB 170:GOTO
   15
60:"Z":PAUSE " ":
   F=(B*B-A*C):IF
   F=0GOTO 150
65:X=U+(B*E-C*D)/
   F:Y=V+(A*E-B*D
   )/F:GOSUB 200
70:T=SGN B*45:IF
   A=CGOTO 80
75:T=.5*ATN (2*B/
   (A-C))
80:G=(C*COS T*COS
   T-2*B*COS T*
   SIN T+A*SIN T*
   SIN T)/-F:G=√G
85:H=(C*SIN T*SIN
   T+2*B*COS T*
   SIN T+A*COS T*
   COS T)/-F:H=√H
   :IF H>=GGOTO 9
   5
90:Z=H:H=G:G=Z:T=
   T+90
95:PRINT "BRG=";J
   :PRINT "RNG=";
   K:GOTO 145
100:"H":PRINT " TP
    E=A EST=Z SIZ=
    S PRB=X"
105:PRINT "CON=C":
    GOTO 100
120:"S":INPUT "SIZ
    E? ";S:O=1-EXP
    (-S*S/2)
125:PRINT USING "#
    #.##";"PRB=";O
    :USING :GOTO 1
    35
130:"X":INPUT "PRB
    ? ";O:S=√(-2*
    LN (1-O)):
    PRINT "SIZE=";
    S
135:X=S*G:Y=S*H:N=
    T:PRINT "SMJ="
    ;Y:IF N<0LET N
    =N+180
140:PRINT "DIR=";N
    :PRINT "SMI=";
    X:PRINT "A=";π
    *X*Y
145:PRINT "END":
    GOTO 145
150:PRINT "NO SOL"
    :GOTO 150
170:X=U-R*SIN Q:Y=
    V-R*COS Q:
    GOSUB 200
175:W=(P-J):L=K*O*
    π/180:G=COS J/
    L:H=SIN J/L:IF
    W>=180LET W=W-
    360:GOTO 185
180:IF W<=-180LET
    W=W+360
185:W=W/O:A=G*G+A:
    B=G*H+B:C=H*H+
    C:D=W*G+D:E=W*
    H+E:RETURN
200:K=√(X*X+Y*Y):
    IF K=0LET J=0:
    RETURN
205:J=ACS (Y/K):IF
    ASN (X/K)<0LET
    J=360-J
210:RETURN
```

SHARP PC-1211 PROGRAM

```
 10:"A":CLEAR :
    DEGREE
 15:"C":INPUT "O
    BS BRG? ";P:
    INPUT "STA B
    RG? ";Q:
    INPUT "STA R
    NG? ";R:
    INPUT "BRG E
    RR? ";O:IF I
    =2GOTO 130
 75:I=I+1:A(I+26
    )=P:A(I+28)=
    Q:A(I+30)=R:
    A(I+32)=O:IF
    I=1GOTO 15
 85:X=A(31)*SIN
    (A(29)-A(27)
    ):Y=A(32)*
    SIN (A(30)-A
    (28)):Z=SIN
    (A(28)-A(27)
    )
 90:IF Z=0GOTO 3
    25
 95:U=(X*SIN A(2
    8)-Y*SIN A(2
    7))/Z:V=(X*
    COS A(28)-Y*
    COS A(27))/Z
105:FOR I=1TO 2:
    P=A(I+26):Q=
    A(I+28):R=A(
    I+30):O=A(I+
    32):GOSUB 40
    0
110:NEXT I:GOTO
    15
130:GOSUB 400
135:GOTO 15
140:"Z":F=(B*B-A
    *C):IF F=0
    GOTO 325
150:X=U+(B*E-C*D
    )/F:Y=V+(A*E
    -B*D)/F:
    GOSUB 500
160:T=SGN B*45:
    IF A=CTHEN 1
    70
165:T=.5*ATN (2*
    B/(A-C))
170:G=(C*COS T*
    COS T-2*B*
    COS T*SIN T+
    A*SIN T*SIN
    T)/-F:G=√G
175:H=(C*SIN T*
    SIN T+2*B*
    COS T*SIN T+
    A*COS T*COS
    T)/-F:H=√H:
    IF H>=GGOTO
    185
180:Z=H:H=G:G=Z:
    T=T+90
185:PRINT "BRG="
    ;J:PRINT "RN
    G=";K:GOTO 3
    20
200:"H":PRINT "T
    PE=A EST=Z S
    IZ=S PRB=X"
205:PRINT "CON=C
    ":GOTO 200
225:"S":INPUT "S
    IZE? ";S:O=1
    -EXP (-S*S/2
    )
230:PRINT USING
    "##.##";"PRB
    =";O:USING :
    GOTO 300
235:"X":INPUT "P
    RB? ";O:S=√(
    -2*LN (1-O))
    :PRINT "SIZE
    =";S
300:X=S*G:Y=S*H:
    N=T:PRINT "S
    MJ=";Y:IF N
    <=0LET N=N+1
    80
315:PRINT "DRC="
    ;N:PRINT "SM
    I=";X:Z=π*X*
    Y:PRINT "A="
    ;Z
320:PRINT "END":
    GOTO 320
325:PRINT "NO SO
    L":GOTO 325
400:X=U-R*SIN Q:
    Y=V-R*COS Q:
    GOSUB 500:W=
    (P-J):L=K*O*
    π/180:G=COS
    J/L:H=SIN J/
    L
415:IF W>=180LET
    W=W-360:GOTO
    440
425:IF W<=-180
    LET W=W+360
440:W=W/O:A=G*G+
    A:B=G*H+B:C=
    H*H+C:D=W*G+
    D:E=W*H+E:
    RETURN
500:K=√(X*X+Y*Y)
    :IF K=0LET J
    =0:RETURN
510:M=ASN (X/K):
    J=ACS (Y/K):
    IF M<0LET J=
    360-J
515:RETURN
```

CASIO FX-702P PROGRAM

```
  5 INP "OPTION",J:
    SET F2:MODE 4
 10 IF J=0 THEN 175
 15 IF J=2 THEN 100
 20 IF J=3 THEN 145
 25 IF J=4 THEN 150
 30 IF J=5 THEN 45
 40 VAC :SET F2
 45 INP "OBS BRG",P
    :INP "STA BRG",
    Q:INP "STA RNG"
    ,R
 50 INP "BRG ERR",O
    :IF I=2 THEN 95
 55 I=I+1:A(I-1)=P:
    A(I+1)=Q:A(I+3)
    =R:A(I+5)=O
 60 IF I=1 THEN 45
 65 X=A(4)*SIN (A(2
    )-A(0)):Y=A(5)*
    SIN (A(3)-A(1))
 70 Z=SIN (A(1)-A(0
    )):IF Z=0 THEN
    170
 75 U=(X*SIN A(1)-Y
    *SIN A(0))/Z:V=
    (X*COS A(1)-Y*C
    OS A(0))/Z
 80 FOR M=0 TO 1:P=
    A(M):Q=A(M+2)
 85 R=A(M+4):O=A(M+
    6):GSB 200:NEXT
     M:GOTO 45
 90 R=A(M+4):O=A(M+
    6):GSB 200:NEXT
     M:GOTO 45
 95 GSB 200:GOTO 45
100 F=(B*B-A*C):IF
    F=0 THEN 170
105 X=U+(B*E-C*D)/F
    :Y=V+(A*E-B*D)/
    F:RPC Y,X
110 T=SGN B*45:IF A
    =C THEN 120
115 T=.5*ATN (2*B/(
    A-C))
120 G=(C*COS T*COS
    T-2*B*COS T*SIN
     T+A*SIN T*SIN
    T)/-F
125 H=(C*SIN T*SIN
    T+2*B*COS T*SIN
     T+A*COS T*COS
    T)/-F
130 G=SQR G:H=SQR H
    :IF G>H;Z=H:H=G
    :G=Z:T=T+90
135 IF Y<0;Y=Y+360
140 PRT "BRG=";Y:PR
    T "RNG=";X:GOTO
     165
145 INP "SIZE",S:O=
    1-EXP (-S*S/2):
    PRT "PRB=";O:GO
    TO 155
150 INP "PRB",O:S=S
    QR (-2*LN (1-O)
    ):PRT "SIZE=";S
155 X=S*G:Y=S*H:PRT
     "SMJ=";Y:N=T:I
    F N<0;N=N+180
160 PRT "DIR=";N:PR
    T "SMI=";X:PRT
    "A=";π*X*Y
165 PRT "END":GOTO
    165
170 PRT "NO SOL":GO
    TO 170
175 PRT "TPE=1 EST=
    2 SIZ=3"
180 PRT "PRB=4 CON=
    5":GOTO 175
200 X=U-R*SIN Q:Y=V
    -R*COS Q:RPC Y,
    X
205 W=(P-Y):L=X*O*π
    /180:G=COS Y/L:
    H=SIN Y/L
210 IF W≥180;W=W-36
    0:GOTO 220
215 IF W≤-180;W=W+3
    60
220 W=W/O:A=G*G+A:B
    =G*H+B:C=H*H+C:
    D=W*G+D:E=W*H+E
    :RET
```

```
000  47 CMS
001  60 DEG
002  02  2
003  42 STO
004  09  09
005  69 OP
006  28  28
007  76 LBL
008  16 A'
009  09  9
010  42 STO
011  00  00
012  42 STO
013  01  01
014  92 RTN
015  76 LBL
016  18 C'
017  69 OP
018  20  20
019  72 ST*
020  00  00
021  92 RTN
022  76 LBL
023  19 D'
024  69 OP
025  21  21
026  73 RC*
027  01  01
028  92 RTN
029  76 LBL
030  10 E'
031  65  ×
032  89  π
033  55  ÷
034  01  1
035  08  8
036  00  0
037  95  =
038  92 RTN
039  76 LBL
040  11  A
041  81 RST
042  76 LBL
043  12  B
044  18 C'
045  91 R/S
046  18 C'
047  91 R/S
048  18 C'
049  91 R/S
050  10 E'
051  18 C'
052  87 IFF
053  03  03
054  01  01
055  28  28
056  19 D'
057  75  -
058  19 D'
059  95  =
060  94 +/-
061  38 SIN
062  65  ×
063  19 D'
064  95  =
065  48 EXC
066  19  19
067  22 INV
068  97 DSZ
069  09  09
070  00  00
071  76  76
072  01  1
073  69 OP
074  21  21
075  92 RTN
076  42 STO
077  18  18
078  65  ×
079  43 RCL
080  14  14
081  42 STO
082  20  20
083  38 SIN
084  75  -
085  43 RCL
086  10  10
087  22 INV
088  44 SUM
089  20  20
090  38 SIN
091  65  ×
092  43 RCL
093  19  19
094  95  =
095  55  ÷
096  43 RCL
097  20  20
098  38 SIN
099  95  =
100  42 STO
101  38  38
102  43 RCL
103  14  14
104  39 COS
105  65  ×
106  43 RCL
107  18  18
108  75  -
109  43 RCL
110  10  10
111  39 COS
112  65  ×
113  43 RCL
114  19  19
115  95  =
116  55  ÷
117  43 RCL
118  20  20
119  38 SIN
120  95  =
121  42 STO
122  39  39
123  86 STF
124  03  03
125  02  2
126  42 STO
127  09  09
128  16 A'
129  19 D'
130  10 E'
131  42 STO
132  18  18
133  19 D'
134  32 X:T
135  19 D'
136  32 X:T
137  37 P/R
138  75  -
139  43 RCL
140  38  38
141  95  =
142  94 +/-
143  32 X:T
144  75  -
145  43 RCL
146  39  39
147  95  =
148  94 +/-
149  32 X:T
```

```
150  22  INV
151  37  P/R
152  42  STO
153  19   19
154  10  E'
155  75   -
156  43  RCL
157  18   18
158  95   =
159  94  +/-
160  32  X:T
161  69  OP
162  21   21
163  64  PD*
164  01   01
165  42  STO
166  18   18
167  89   π
168  32  X:T
169  22  INV
170  77   GE
171  01   01
172  78   78
173  75   -
174  32  X:T
175  65   ×
176  02   2
177  95   =
178  49  PRD
179  18   18
180  73  RC*
181  01   01
182  35  1/X
183  42  STO
184  27   27
185  42  STO
186  26   26
187  32  X:T
188  43  RCL
189  19   19
190  37  P/R
191  42  STO
192  28   28
193  49  PRD
194  27   27
195  33  X²
196  44  SUM
197  23   23
198  43  RCL
199  18   18
200  49  PRD
201  26   26
202  49  PRD
203  27   27
204  32  X:T
205  49  PRD
206  28   28
207  49  PRD
208  26   26
209  33  X²
210  44  SUM
211  21   21
212  43  RCL
213  28   28
214  44  SUM
215  22   22
216  43  RCL
217  26   26
218  44  SUM
219  24   24
220  43  RCL
221  27   27
222  44  SUM
223  25   25
224  22  INV
225  97  DSZ
226  09   09
227  02   02
228  32   32
229  61  GTO
230  01   01
231  29   29
232  69  OP
233  28   28
234  43  RCL
235  08   08
236  92  RTN
237  76  LBL
238  13   C
239  43  RCL
240  21   21
241  42  STO
242  41   41
243  43  RCL
244  22   22
245  42  STO
246  42   42
247  43  RCL
248  23   23
249  42  STO
250  43   43
251  42  STO
252  10   10
253  65   ×
254  43  RCL
255  21   21
256  22  INV
257  44  SUM
258  10   10
259  75   -
260  43  RCL
261  22   22
262  22  INV
263  49  PRD
264  10   10
265  33  X²
266  95   =
267  35  1/X
268  49  PRD
269  41   41
270  49  PRD
271  42   42
272  49  PRD
273  43   43
274  02   2
275  55   ÷
276  43  RCL
277  10   10
278  95   =
279  24  CE
280  22  INV
281  30  TAN
282  55   ÷
283  02   2
284  95   =
285  42  STO
286  29   29
287  43  RCL
288  43   43
289  65   ×
290  43  RCL
291  24   24
292  75   -
293  43  RCL
294  42   42
295  65   ×
296  43  RCL
297  25   25
298  85   +
299  43  RCL
```

300	38	38
301	95	=
302	32	X:T
303	43	RCL
304	42	42
305	65	×
306	43	RCL
307	24	24
308	75	-
309	43	RCL
310	41	41
311	65	×
312	43	RCL
313	25	25
314	85	+
315	43	RCL
316	39	39
317	95	=
318	32	X:T
319	22	INV
320	37	P/R
321	42	STO
322	32	32
323	01	1
324	32	X:T
325	42	STO
326	33	33
327	43	RCL
328	29	29
329	37	P/R
330	42	STO
331	11	11
332	33	X²
333	42	STO
334	12	12
335	42	STO
336	13	13
337	32	X:T
338	49	PRD
339	11	11
340	33	X²
341	42	STO
342	14	14
343	42	STO
344	15	15
345	43	RCL
346	41	41
347	49	PRD
348	12	12
349	49	PRD
350	14	14
351	43	RCL
352	42	42
353	65	×
354	02	2
355	95	=
356	49	PRD
357	11	11
358	43	RCL
359	43	43
360	49	PRD
361	13	13
362	49	PRD
363	15	15
364	43	RCL
365	15	15
366	85	+
367	43	RCL
368	11	11
369	85	+
370	43	RCL
371	12	12
372	95	=
373	34	√X
374	42	STO
375	16	16
376	43	RCL
377	13	13
378	75	-
379	43	RCL
380	11	11
381	85	+
382	43	RCL
383	14	14
384	95	=
385	34	√X
386	42	STO
387	17	17
388	43	RCL
389	32	32
390	91	R/S
391	43	RCL
392	33	33
393	92	RTN
394	61	GTO
395	03	03
396	93	93
397	76	LBL
398	15	E
399	75	-
400	01	1
401	95	=
402	94	+/-
403	23	LNX
404	65	×
405	02	2
406	95	=
407	94	+/-
408	34	√X
409	91	R/S
410	42	STO
411	15	15
412	65	×
413	43	RCL
414	16	16
415	95	=
416	42	STO
417	18	18
418	32	X:T
419	43	RCL
420	29	29
421	94	+/-
422	42	STO
423	30	30
424	43	RCL
425	15	15
426	65	×
427	43	RCL
428	17	17
429	95	=
430	77	GE
431	04	04
432	41	41
433	48	EXC
434	18	18
435	32	X:T
436	09	9
437	00	0
438	44	SUM
439	30	30
440	32	X:T
441	91	R/S
442	32	X:T
443	43	RCL
444	30	30
445	91	R/S
446	43	RCL
447	18	18
448	91	R/S
449	65	×

450	32	X:T
451	65	×
452	89	π
453	95	=
454	92	RTN
455	61	GTO
456	04	04
457	54	54
458	76	LBL
459	14	D
460	33	X^2
461	55	÷
462	32	X:T
463	02	2
464	95	=
465	94	+/-
466	22	INV
467	23	LNX
468	75	-
469	01	1
470	95	=
471	94	+/-
472	91	R/S
473	32	X:T
474	34	$\sqrt{X}$
475	61	GTO
476	04	04
477	10	10
478	00	0
479	00	0

Appendix 1. The Estimation Procedure

In this development the assumptions stated in Section I are required conditions. A rectangular coordinate system is used with the positive y-axis directed north, the positive x-axis directed east and the origin at the reference point and all angles are in radians. Figure 3 shows a station located with respect to the coordinate system. The bearing line of length r goes to the object's unknown position, the bearing line of length r goes to an initial estimate of the object's position and the third bearing line corresponds to an observed bearing.

To determine the coordinates for a final estimate, consider the arc segments $u = r(\theta-\phi)$ where $\theta-\phi$ is the bearing error and $v = r(\phi-\beta)$ and $w = r(\theta-\beta)$ that are defined by the three bearing lines and the circle of radius r that is centered on the station and goes through the initial estimate. The geometry is shown in Figure 3. Note that u can be defined by $u = w - v$. In this expression, $w = r(\theta-\beta)$ is known, and v can be determined in terms of x and y the unknown coordinates of the target. With the reference point at the initial estimte, to first order, $v = x \cos \beta - y \sin \beta$ and $u = w - x \cos \beta + y \sin \beta$. Since this approximation applies to all stations, its use suggests that, for each station, $\mathit{r} \simeq r$ which is equivalent to having the initial estimate relatively close to the target's position.

Since, for each station i, an observed bearing θ_i is the value of a normal random variable Θ_i with mean ϕ_i, the coordinate $u_i = r_i(\theta_i-\phi_i)$ is the value of a normal random variable U_i with mean zero. In addition, since the Θ_i are independent, the U_i are

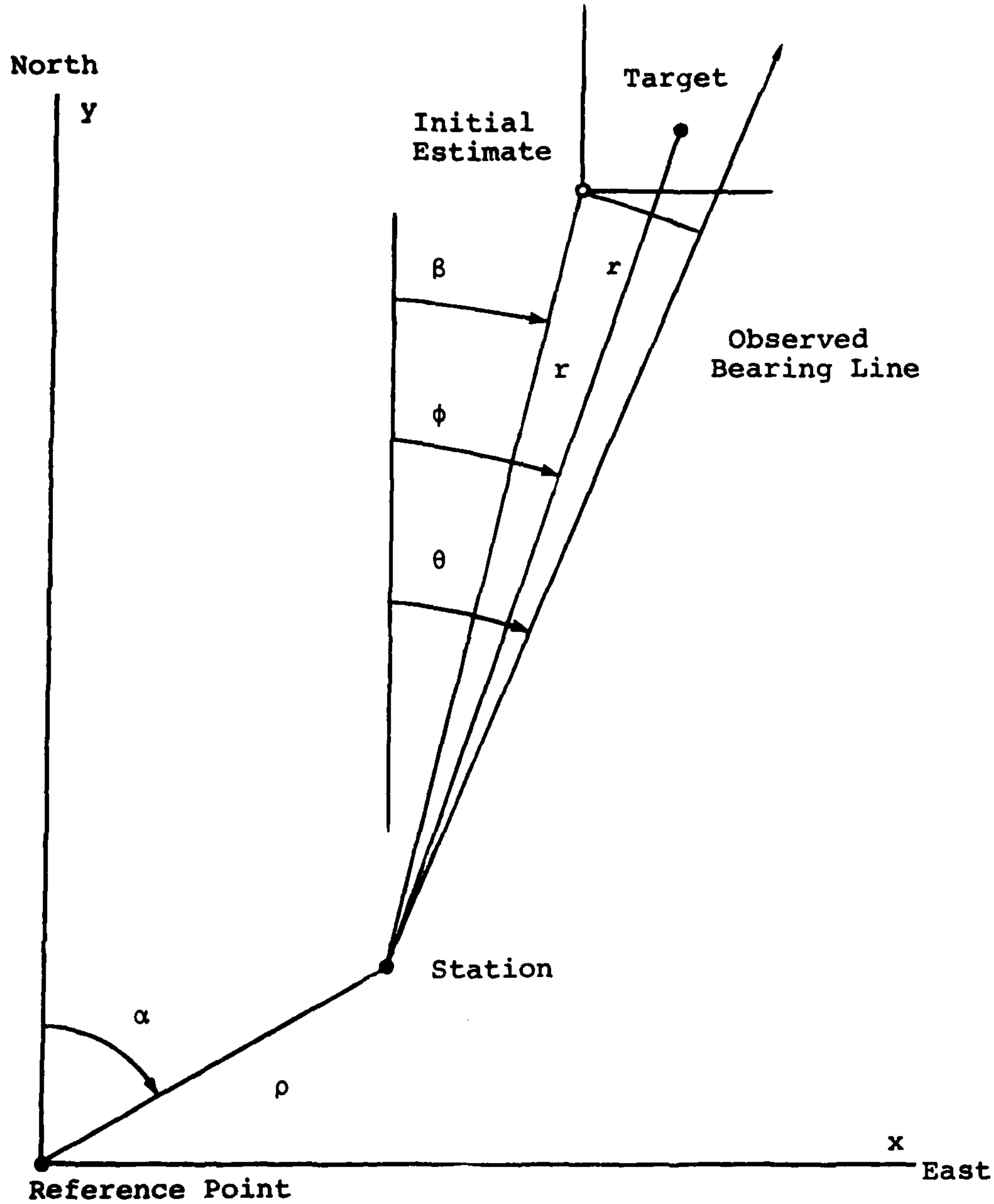

Figure 3. The coordinate geometry. The coordinates of the initial estimate are (x^*, y^*). In the development, the reference point is at the initial estimate.

also independent. And, consequently, the joint distribution of the U_i is determined.

To estimate x and y, maximum likelihood estimates are used here. The likelihood for a sample $\theta_i, \theta_2, \ldots, \theta_n$ from n stations is

$$L(\theta_1,\theta_2,\ldots,\theta_n) = \prod_1^n \frac{1}{\sqrt{2\pi}\, e_i} \exp - \frac{1}{2} \sum_1^n (\theta_i - \phi_i)^2/e_i^2$$

and the likelihood for a cooresponding sample $u_1, u_2, \ldots, u_n$ is

$$L(u_1,u_2,\ldots,u_n) = \prod_1^n \frac{1}{\sqrt{2\pi}\, \sigma_i} \exp - \frac{1}{2} \sum_1^n u_i^2/\sigma_i^2$$

with $\sigma_i = r_i e_i$ where e_i is the standard deviation of Θ_i.

By definition, the maximum likelihood estimates for x and y are the estimates which make $L(u_1, u_2, \ldots, u_n)$ a maximum. In this case, making $L(u_1,u_2,\ldots,u_n)$ a maximum is equivalent to making $\sum_1^n (u_i{}^2/\sigma_i{}^2)$ a minimum. So, to find the maximum likelihood estimates $\hat{x}$ and $\hat{y}$, solve the following two equations for $\hat{x}$ and $\hat{y}$:

$$\left.\frac{\partial(\ln L)}{\partial x}\right|_{x=\hat{x}} = 0 \quad \text{and} \quad \left.\frac{\partial(\ln L)}{\partial y}\right|_{y=\hat{y}} = 0$$

With $u_i = w_i - x \cos \beta_i + y \sin \beta_i$, the equations can be written as follows:

$$\sum_1^n (w_i - \hat{x} \cos \beta_i + \hat{y} \sin \beta_i)(\cos \beta_i)/\sigma_i^2 = 0$$

and

$$\sum_1^n (w_i - \hat{x} \cos \beta_i + \hat{y} \sin \beta_i)(\sin \beta_i)/\sigma_i^2 = 0 .$$

In terms of the following quantities:

$A = \Sigma(\cos^2 \beta_i)/\sigma_i^2$, $B = \Sigma(\sin \beta_i \cos \beta_i)/\sigma_i^2$,

$C = \Sigma(\sin^2 \beta_i)/\sigma_i^2$, $D = \Sigma(w_i \cos \beta_i)/\sigma_i^2$,

$E = \Sigma(w_i \sin \beta_i)/\sigma_i^2$,

the equations become:

$$A\hat{x} - B\hat{y} = D$$

$$B\hat{x} - C\hat{y} = E$$

The solutions are:

(1) $$\hat{x} = (BE - CD)/(B^2 - AC)$$

and

(2) $$\hat{y} = (AE - BD)/(B^2 - AC)$$

A confidence region can be constructed about an estimated position. In order to indicate how this can be done, a probability region about the true position will be considered first.

Both $\hat{x}$ and $\hat{y}$ are values of random variables. If a new set of bearings $\theta_1, \theta_2, \ldots, \theta_n$ is observed (for the same initial estimate and a fixed target), in general, a new pair of values $\hat{x}$ and $\hat{y}$ will be obtained.

If $\hat{X}$ and $\hat{Y}$ represent these random variables,

$$\hat{X} = \frac{1}{(B^2-AC)} \sum_{1}^{n} (W_i/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)$$

$$\hat{Y} = \frac{1}{(B^2-AC)} \sum_{1}^{n} (W_i/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)$$

where $W_i = r_i(\Theta_i - \beta_i)$.

Since $\hat{X}$ and $\hat{Y}$ are a linear combination of the n normal random variables $W_1, W_2, \ldots, W_n$, or equivalently of the n normal random variables $\Theta_1, \Theta_2, \ldots, \Theta_n$, they have a joint normal distribution. Since $E(W_i) = r_i(\phi_i - \beta_i)$, if $\beta_i = \phi_i$ for $i = 1, 2, \ldots, n$, that is, if the initial estimate of the target's position is at the target's position, $E(W_1) = 0$ for $i = 1, 2, \ldots, n$. In this case $E(\hat{X}) = 0$ and $E(\hat{Y}) = 0$ and the joint normal distribution is centered on the object's position. To the degree of the approximations that have been made, this is also true if the initial estimate is not at the target's position.

A region of minimum area for a given probability of containment of an estimated position can be determined. The region is bounded by an ellipse which is centered on the object's position and whose axes lie along the axes of an x'y'-coordinate system that is obtained by rotating the xy-coordinate system that is centered on the object's position through an angle γ. In this system, $\sigma_{\hat{x}',\hat{y}'}$ is 0, that is $\hat{x}'$ and $\hat{y}'$ are independent normal random variables. The two coordinate systems are illustrated in

Figure 4. The coordinates of a point in the two systems are related by

$$x' = x \cos \gamma - y \sin \gamma$$

$$y' = x \sin \gamma + y \cos \gamma$$

These relations imply:

$$(3) \qquad \sigma^2_{\hat{x}'} = \sigma^2_{\hat{x}} \cos^2 \gamma - 2\sigma_{\hat{x}\hat{y}} \cos \gamma \sin \gamma + \sigma^2_{\hat{y}} \sin^2 \gamma \ ,$$

$$(4) \qquad \sigma^2_{\hat{y}'} = \sigma^2_{\hat{x}} \sin^2 \gamma + 2\sigma_{\hat{x}\hat{y}} \cos \gamma \sin \gamma + \sigma^2_{\hat{y}} \cos^2 \gamma$$

and

$$(5) \qquad \sigma^2_{\hat{x}'\hat{y}'} = (\sigma^2_{\hat{x}} - \sigma^2_{\hat{y}}) \sin \gamma \cos \gamma + \sigma_{\hat{x}\hat{y}} (\cos^2 \gamma - \sin^2 \gamma)$$

where γ, the angle of rotation of the coordinate axes, is positive in the clockwise direction. And $\sigma_{\hat{x}'\hat{y}'} = 0$ implies

$$\tan 2\gamma = \frac{2\sigma_{\hat{x}\hat{y}}}{\sigma^2_{\hat{y}} - \sigma^2_{\hat{x}}}$$

With the initial estimate of the target's position at the target's position $E(W_i) = 0$ and therefore, $Var(W_i) = \sigma_i{}^2$ for $i = 1, 2, \ldots, n$. In this case

$$\sigma^2_{\hat{x}} = \frac{1}{(B^2 - AC)^2} \sum_{1}^{n} (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)^2,$$

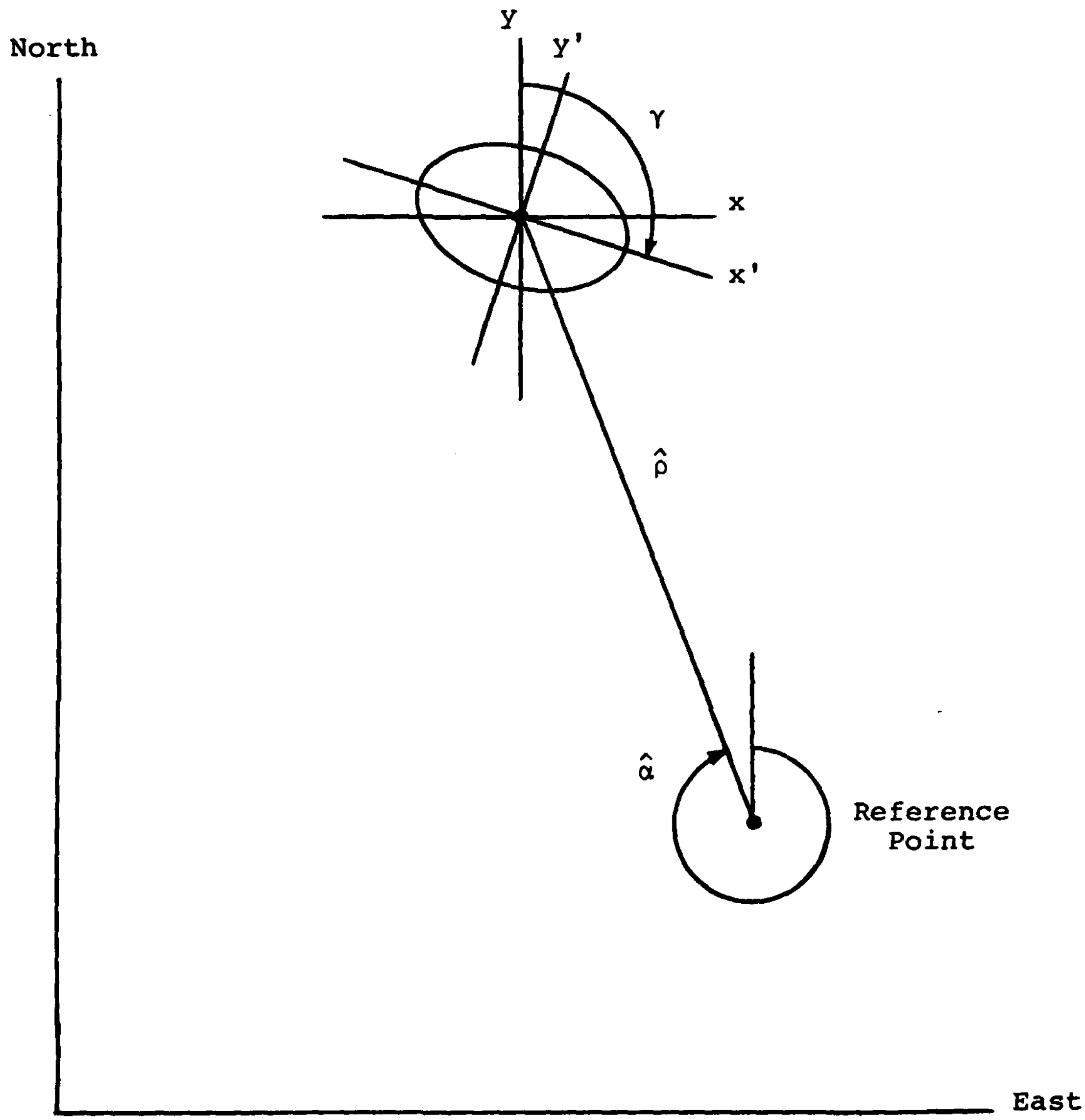

Figure 4. An elliptical confidence region and the primed coordinate system in which the covariance $\sigma_{\hat{x}'\hat{y}'}$ is zero. The center of the ellipse and the origin of the coordinate systems are at the target's estimated position. The estimated bearing $\hat{\alpha}$ and estimated range $\hat{\rho}$ are indicated for a reference position.

$$\sigma^2_{\hat{y}} = \frac{1}{(B^2-AC)^2} \sum_1^n (1/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)^2$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{1}{(B^2-AC)} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)(A \sin \beta_i - B \cos \beta_i).$$

Using the definition for A, B and C, the above become

(6) $$\sigma^2_{\hat{x}} = \frac{C}{(AC-B^2)} ,$$

(7) $$\sigma^2_{\hat{y}} = \frac{A}{(AC-B^2)} ,$$

and

(8) $$\sigma_{\hat{x}\hat{y}} = \frac{B}{(AC-B^2)} .$$

So tan 2 $= 2B/(A-C)$ for $\beta_i = \phi_i$, $i = 1, 2, \ldots, n$.

With the target's position known and, consequently, ϕ_i known for $i = 1, 2, \ldots, n$, the above expressions for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be used, since the initial estimate of the target's position can be taken as the target's position.

With values for $\sigma_{\hat{x}}$, $\sigma_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ, values for $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$ can be found by using equations (3) and (4). The probability that an estimated position will be within an ellipse of semiaxes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ which is centered on the target's position is

$1 - \exp(-k^2/2)$. This result can be found by integrating the bivariate normal density over the ellipse. And the area of the ellipse is $k^2 \sigma_{\hat{x}'} \sigma_{\hat{y}'}$.

Given estimates $\hat{x}$ and $\hat{y}$ found by using Equations (1) and (2), the ellipse with semi-axes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ in a coordinate system that is centered on the point $(\hat{x},\hat{y})$ and has been rotated through an angle γ is a $1 - \exp(-k^2/2)$ confidence region. This follows, since, to the degree of the approximations involved, the bivariate normal distribution of X and Y is centered on the target's position. The confidence ellipse is defined if $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$ and $\sigma_{\hat{x}\hat{y}}$ can be found, that is if the elements of the covariance matrix can be found. To the degree of the approximations involved, this can be done as follows: First, assume the initial estimate of the target's position is at the target's position. Then, values for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be determined by using Equations (6), (7) and (8). These values can then be used to determine $\sigma^2_{\hat{x}'}$, $\sigma^2_{\hat{y}'}$ and $\sigma_{\hat{x}'\hat{y}'}$ by using Equations (3), (4) and (5). Now, with a value for k, a confidence region can be constructed. To the degree of the approximations involved, the shape of the confidence region is independent of both the target's position and of the initial estimate of the target's position.

For the case where bearings are taken from the target on two or more stations, θ_i is the reciprocal of the bearing taken from the target.

A discussion of the theory of bearings only position estimation procedures for situations similar to the one considered here is given in Reference 1. Reference 2 gives an essentially equivalent bearings only procedure. It also gives a range only procedure, a range and bearing procedure and HP-9830A programs with which to implement the procedures. Using the fix determined by two lines of bearing as the initial estimate was suggested by this reference.

The following equations are evaluated in the program to determine the coordinates of the initial estimate:

$$x^* \sin(\theta_2-\theta_1) = [\rho_1 \sin(\alpha_1-\theta_1)] \sin\theta_2 - [\rho_2 \sin(\alpha_2-\theta_2)] \sin\theta_1$$

and

$$y^* \sin(\theta_2-\theta_1) = [\rho_1 \sin(\alpha_1-\theta_1)] \cos\theta_2 - [\rho_2 \sin(\alpha_2-\theta_2)] \cos\theta_1 .$$

Reference 3 describes a TI-59 program that is based on an equivalent procedure. The program allows a user to either input the coordinates of the initial estimate or determine them in the manner described here.

Appendix 2. HP-41CV Program Labels

The global labels in the HP-41CV program that are assigned to the keys Σ+, 1/x, √x, LOG and LN give these keys a mnemonic character. For example, with the calculator in USER mode, if LOG is pressed and held, SIZ will be displayed and then after a delay, NULL.

If a global label is replaced by a local label, this mnemonic character will be lost. However, automatic key assignment will be gained if the key corresponding to the local label has not been previously assigned. If this is the case and if there is a second program in program memory that uses the same local label, then it will be automatically assigned when the calculator is positioned in program memory at that program.

Global labels in the HP-41CV program can be replaced with local labels. If this is done as described below, the user instructions will still be applicable. First, make the following replacements:

Line 329: LBL "PRB" → LBL E. Line 311: LBL "SIZ" → LBL D.
Line 186: LBL "EST" → LBL B. Line 008: LBL "CON" → LBL C.
Line 047: GTO "CON" → GTO C.

Next, after line 001: LBL "TPE", insert LBL A so that it becomes line 002. Finally, if there is a key assignment for a key in the following list: Σ+, 1/x, √x, LOG and LN, remove it. Now, with the calculator in USER mode, if LOG is pressed and held, XEQ D will be displayed and then, after a delay, NULL. The key's mnemonic character has been lost, but if there is a second program in program memory with the local label D, LOG will be automatically assigned to D when the calculator is positioned in program memory at that program.

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